# A weighted version of quantization commutes with reduction for a toric manifold

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ABSTRACT. We compute explicitly the equivariant Hirzebruch $\chi_y$-characteristic of an equivariant complex line bundle over a toric manifold and state a weighted version of the quantization commutes with reduction principle in symplectic geometry. Then, we give a weighted decomposition formula for any simple polytope in $\mathbb{R}^n$. This formula generalizes a polytope decomposition due to Lawrence [**10**] and Varchenko [**14**] and extends a previous weighted version obtained by Karshon, Sternberg and Weitsman [**9**].

## 1. Introduction

A convex polytope is a convex bounded polyhedron in $\mathbb{R}^n$. We say that a polytope is simple if there are exactly $n$ edges emanating from each of its vertices. In addition, if the lines along the edges of a simple polytope are generated by a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^n$, the polytope is called regular.

Let $M$ be a toric manifold and let T be the torus acting on $M$. The image of $M$ under the moment map of the torus action is a regular polytope $\Delta$ which lies on $\mathfrak{t}^*$ (the dual of the Lie algebra $\mathfrak{t}$ of T). Each integral lattice point in $\mathfrak{t}^*$ corresponds to a character of a representation of T. Counting lattice points in $\Delta$ is closely related to the quantization commutes with reduction principle. For the Dolbeault quantization, this principle states that each character $\alpha$ of T occurs in the quantization of $M$ with multiplicity 1 or 0 depending on whether the lattice point representing $\alpha$ is in $\Delta$ or not. (c.f. [**5**], [**6**].)

We use the Atiyah-Bott Berline-Vergne localization formula in equivariant cohomology and a notion of polarization to give a weighted version of the quantization commutes with reduction principle for a toric manifold. This is expressed as a way of counting lattice points in $\Delta$ with *weights*. From a combinatorial point of view, we can count lattice points in $\Delta$ using a polytope decomposition due to Lawrence [**10**] and Varchenko [**14**]. (See Figure 1 in Subsection §§2.1.) We give a weighted decomposition formula for any simple polytope that generalizes the Lawrence and Varchenko polytope decomposition. This formula also extends a previous weighted version obtained by Karshon, Sternberg and Weitsman. (See [**8**] and [**9**].)

## 2. Definitions and Facts

For the reader's convenience, we collect in this section some definitions, facts and remarks to be used later on in this paper.

**2.1. Ordinary polar decomposition.** A convex **polytope** $\Delta$ in $\mathbb{R}^n$ is a **convex bounded polyhedron** given by an intersection of half-spaces $\Delta = \cap H_i$, where

$$H_i = \{x \in \mathbb{R}^n \,|\, \langle x, u_i \rangle \geq \lambda_i \,, \lambda_i \in \mathbb{R}\} \qquad \text{for } i = 1, \cdots, N > n,$$

with $u_i \in (\mathbb{R}^n)^* \simeq \mathbb{R}^n$ and where $N$ denotes the number of $(n-1)$-dimensional faces (the facets) of $\Delta$. The vector $u_i$ can be thought of as the inward normal to the $i$-th facet of $\Delta$ contained in the hyperplane $\langle x, u_i \rangle = \lambda_i$. We assume that $\Delta$ is obtained with the smallest possible $N$. We also assume that our polytopes are always compact and convex.

A polytope is **simple** if there are exactly $n$ edges emanating from each vertex and is **regular**[1] if, in addition, the edges emanating from each vertex lie along lines which are generated by an integral basis of $\mathbb{Z}^n$. If the vertices of a polytope are in the lattice $\mathbb{Z}^n$, we call it **integral**.

Let $\Delta$ be a simple polytope in $\mathbb{R}^n$. Then, we have $n$ edges emanating from each of its vertices. Let $v$ be a vertex. We choose $\alpha_{1,v}, \ldots, \alpha_{n,v}$ in the direction of the edges pointing from $v$ to other vertices of $\Delta$. Up to a positive scalar and permutation, these *edge vectors* are uniquely determined. When $\Delta$ is regular and integral, we can choose $\alpha_{1,v}, \ldots, \alpha_{n,v}$ to have integer entries. In this case, we shall impose the additional normalization condition that the edge vectors be primitive elements of the lattice $\mathbb{Z}^n$; namely, none of them can be expressed as a multiple of a lattice element by an integer greater than one. This fixes our choice of edge vectors for such polytopes.

The tangent cone $\mathbf{C}_v$ at $v$ is

$$\mathbf{C}_v = \{v + r(x - v) \,|\, r \geq 0, x \in \Delta\} = v + \sum_{j=1}^{n} \mathbb{R}_{\geq 0}\alpha_{j,v}.$$

Notice that we can always find a vector $\xi$ such that $\langle \alpha_{j,v}, \xi \rangle \neq 0$ for all $v$ and $j$. To see this, we can think of an edge vector as the normal vector of a hyperplane in $\mathbb{R}^n$ passing through the origin. Since there are a finite number of edge vectors, we have a finite number of such hyperplanes. They split $\mathbb{R}^n$ into several connected components. We pick $\xi$ such that it is not perpendicular to the boundary of these components.

DEFINITION 1. A **polarizing vector** $\xi$ for a simple polytope $\Delta \subset \mathbb{R}^n$ is a vector in $\mathbb{R}^n$ such that $\langle \alpha_{j,v}, \xi \rangle \neq 0$ for all vertices $v$ and all edge vectors $\alpha_{1,v}, \ldots, \alpha_{n,v}$.

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[1]our regular polytopes are also known as smooth or Delzant polytopes [**4**], [**6**].

At each vertex $v$, we define the polarized vectors $\alpha_{j,v}^{\sharp}$ by

(1) $$\alpha_{j,v}^{\sharp} = \left\{ \begin{array}{ll} \alpha_{j,v} & \text{if } \langle \alpha_{j,v}, \xi \rangle < 0 \quad \text{(unflipped)} \\ -\alpha_{j,v} & \text{if } \langle \alpha_{j,v}, \xi \rangle > 0 \quad \text{(flipped)} \end{array} \right. .$$

This process is called **polarization**. Then, the polarized tangent cone $\mathbf{C}_v^{\sharp}$ at $v$ is

$$\mathbf{C}_v^{\sharp} = v + \sum_{j=1}^{n} \mathbb{R}_{\geq 0} \alpha_{j,v}^{\sharp}.$$

We have the following (adapted) decomposition of a simple polytope due to Lawrence [**10**] and Varchenko [**14**].

PROPOSITION 1. *Let $\Delta$ be a simple polytope in $\mathbb{R}^n$. For any choice of polarizing vector, we have*

(2) $$\mathbf{1}_{\Delta} = \sum_{v}(-1)^{\#v}\mathbf{1}_{\mathbf{C}_v^{\sharp}},$$

*where the sum is over all vertices $v$ of the polytope and the symbols $\mathbf{1}_{\Delta}$ and $\mathbf{1}_{\mathbf{C}_v^{\sharp}}$ denote the ordinary characteristic functions over $\Delta$ and over the polarized tangent cones $\mathbf{C}_v^{\sharp}$ (possibly with some facets removed). In this formula, $\#v$ denotes the number of edge vectors at $v$ flipped by the polarization process* (1).

Lawrence [**10**] and Varchenko [**14**] formulas actually hold for any simple convex polyhedron of full dimension in $\mathbb{R}^n$. In the case of lattice polytopes, $\Delta \cap \mathbb{Z}^n$, we avoid removing certain facets from the cones $\mathbf{C}_v^{\sharp}$, by *shifting* some of the cones. We illustrate this decomposition in Figure 1.
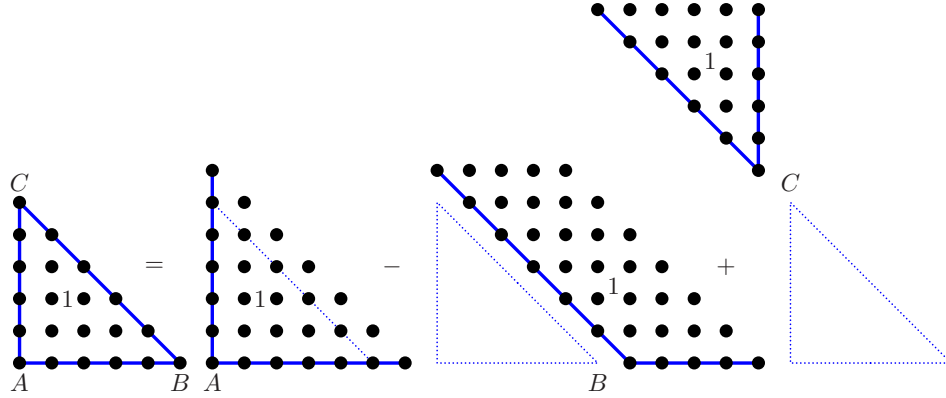


FIGURE 1. Ordinary polar decomposition of a lattice triangle

**2.2. Some classical formal power series.** We will pay attention to three formal power series.

The Todd function $\mathbf{Td}(x)$ is defined as

$$\mathbf{Td}(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots .$$

Since

$$\mathbf{Td}(-x) = \frac{x}{e^x - 1} = e^{-x}\mathbf{Td}(x),$$

we can formally average $\mathbf{Td}(x)$ and $\mathbf{Td}(-x)$ to get

$$
\begin{aligned}
\frac{1}{2}(\mathbf{Td}(x) + \mathbf{Td}(-x)) &= \frac{1}{2}\mathbf{Td}(x)(1 + \mathrm{e}^{-x}) = (x/2)\frac{1 + \mathrm{e}^{-x}}{1 - \mathrm{e}^{-x}} \\
&= (x/2)\frac{\mathrm{e}^{x/2} + \mathrm{e}^{-x/2}}{\mathrm{e}^{x/2} - \mathrm{e}^{-x/2}} = \frac{x/2}{\tanh(x/2)},
\end{aligned}
$$

which is known as the $\hat{\mathbf{L}}(x)$ function. Both $\mathbf{Td}(x)$ and $\hat{\mathbf{L}}(x)$ are particular instances of the more general formal power series

$$
Q(y, x) = \frac{x(1 + y)}{1 - \mathrm{e}^{-x(1+y)}} - yx = \frac{x(1 + y\mathrm{e}^{-x(1+y)})}{1 - \mathrm{e}^{-x(1+y)}}.
$$

This is the Hirzebruch function. When $y = 0$ we get $\mathbf{Td}(x)$ and if $y = 1$ and $x$ is replaced by $x/2$, we get $\hat{\mathbf{L}}(x)$. The Hirzebruch function can also be written as

$$
Q(y, x) = \frac{1}{1 + y}\mathbf{Td}(x(1 + y)) + \frac{y}{1 + y}\mathbf{Td}(-x(1 + y)).
$$

We replace $x$ by $x/(1 + y)$ above and denote the resulting formula by $\mathbf{Q}_y(x)$. We have a weighted average of $\mathbf{Td}(x)$ and $\mathbf{Td}(-x)$. For any $y \neq -1$ it follows that

$$
\mathbf{Q}_y(x) = \frac{1}{1 + y}\mathbf{Td}(x)(1 + y\mathrm{e}^{-x}).
$$

These classical formal power series define well known topological invariants; namely, characteristic classes associated to vector bundles over a manifold. Let $M$ be a compact manifold and let $E \to M$ be a complex vector bundle over $M$. By the Splitting principle in topology (c.f. [**2**]), we can always assume that $E$ splits into a direct sum of complex line bundles,

$$
E = L_1 \oplus \cdots \oplus L_r.
$$

Then the Todd class $\mathbf{Td}(E)$ of $E$ is given by

$$
(3) \qquad\qquad \mathbf{Td}(E) = \prod_{k=1}^{r} \frac{c_1(L_k)}{1 - \mathrm{e}^{-c_1(L_k)}},
$$

where $c_1(L_1), \ldots, c_1(L_r)$ are the first Chern classes of the line bundles $L_1, \ldots, L_r$. Similarly, the $\hat{\mathbf{L}}$-class of $E$ and more generally the Hirzebruch $\mathbf{Q}_y$-class of $E$ are respectively defined by

$$
\hat{\mathbf{L}}(E) = \mathbf{Td}(E) \prod_{k=1}^{r} \frac{1}{2}(1 + \mathrm{e}^{-c_1(L_k)}) \quad \text{and}
$$

$$
(4)
$$

$$
\mathbf{Q}_y(E) = \mathbf{Td}(E) \prod_{k=1}^{r} \frac{1}{1 + y}(1 + y\mathrm{e}^{-c_1(L_k)}).
$$

We say that a bundle $E \to M$ stably splits if $E \oplus \mathbb{C}^r \simeq \oplus L_j$, where $\mathbb{C}^r$ denotes a trivial bundle over $M$ and $\oplus L_j$ is a direct sum of complex line bundles $L_j \to M$. Since $E \simeq \oplus L_k$ (by the Splitting principle) and $\mathbf{Td}(\mathbb{C}^r) = 1$, it follows from (3) that $\mathbf{Td}(E \oplus \mathbb{C}^r) = \mathbf{Td}(E)$. The same is true for $\mathbf{L}(E)$ and $\mathbf{Q}_y(E)$. Now, let T be a torus and let $E$ be a T-equivariant vector bundle over $M$. We have the

corresponding T-equivariant versions of (3) and (4),

$$\mathbf{Td}^{\mathrm{T}}(E) = \prod_{k=1}^{r} \frac{c_1^{\mathrm{T}}(L_k)}{1 - \mathrm{e}^{-c_1^{\mathrm{T}}(L_k)}} \quad \text{and}$$

(5)

$$\mathbf{Q}_y^{\mathrm{T}}(E) = \mathbf{Td}^{\mathrm{T}}(E) \prod_{k=1}^{r} \frac{1}{1+y}(1 + y\mathrm{e}^{-c_1^{\mathrm{T}}(L_k)}).$$

**2.3. Toric manifolds and quantization commutes with reduction.** Let $\mathrm{T} = (S^1)^n$ be a torus whose Lie algebra is denoted by $\mathfrak{t}$. A **toric manifold** is a space $(M, \omega, \mu)$, where $(M, \omega)$ is a compact and connected symplectic $2n$-dimensional manifold on which T acts effectively and in a Hamiltonian fashion with moment map $\mu$. This means that the action of T on $M$ has trivial kernel, that the torus action leaves invariant the symplectic form $\omega$ and that $M$ is provided with a T-equivariant map $\mu : M \to \mathfrak{t}^*$ such that $\imath_{X_\eta}\omega = d\langle\mu, \eta\rangle$ for any $\eta \in \mathfrak{t}$. The function $\langle\mu, \eta\rangle$ is the component of $\mu$ in the direction of $\eta$ and $X_\eta$ is the vector field on $M$ generated by $\eta$. Examples of toric manifolds are the complex projective spaces $\mathbb{C}\mathrm{P}^n$ and blow ups, $\mathbb{C}\mathrm{P}^n\#\overline{\mathbb{C}\mathrm{P}^n}\#\ldots\#\overline{\mathbb{C}\mathrm{P}^n}$.

Under the moment map, toric manifolds (smooth toric varieties) correspond to regular polytopes and toric orbifolds (singular toric varieties) correspond to simple polytopes, modulo equivariant symplectomorphisms and translations respectively (c.f. [**4**], [**5**], [**12**].) The image of a toric manifold under the moment map is called moment polytope and its vertices correspond to the fixed point set of the torus action over the manifold. In general, we say that a symplectic manifold $(M, \omega)$ is **prequantizable** if there is a Hermitian complex line bundle $\mathbb{L} \to M$ such that $c_1(\mathbb{L}) = [\omega]$. Such a line bundle is a **prequantization** of $(M, \omega)$. In the case of a toric manifold, $M$ is prequantizable if and only if the vertices of its moment polytope (possibly shifted) are integral lattice points. (c.f. [**5**], [**6**].)

Let $(M, \omega, \mu)$ be a toric manifold and let $\mathbb{L} \to M$ be a prequantization of $M$. The theory of geometric quantization associates a virtual representation $\mathcal{Q}(M)$ of T to $M$ together with $\mathbb{L}$. The virtual vector space $\mathcal{Q}(M)$ is called the **quantization** of the action of T on $(M, \mathbb{L})$. For instance, for the Dolbeault quantization we have $\mathcal{Q}(M) = \mathrm{kernel}\, D_{\mathbb{L}} \oplus -\mathrm{cokernel}\, D_{\mathbb{L}}$, where $D_{\mathbb{L}}$ is the Dolbeault operator defined on the complex $\Omega^*(M)$ of differential forms on $M$ twisted by $\mathbb{L}$. (c.f. [**5**], [**6**].) In this case, the dimension of $\mathcal{Q}(M)$ can be computed by the Atiyah-Singer index theorem,

(6)
$$\dim \mathcal{Q}(M) := \mathrm{ind}(D_{\mathbb{L}}) = \int_M \mathrm{e}^{c_1(\mathbb{L})}\mathbf{Td}(M),$$

where $\mathbf{Td}(M)$ is the Todd class of the tangent bundle of $M$ and $c_1(\mathbb{L})$ is the first Chern class of the prequantization line bundle $\mathbb{L} \to M$.

The integer $\dim \mathcal{Q}(M)$ in (6) is equal to the number of integral lattice points in $\Delta = \mu(M)$ [**3**]. This is a manifestation of the **quantization commutes with reduction** principle in symplectic geometry. To be more precise, if $\alpha$ is a weight of the torus representation on $\mathcal{Q}(M)$, we define $\mathcal{Q}(M)^\alpha$ as the subspace of $\mathcal{Q}(M)$ on which T acts via the character given by $\alpha$. The *quantization commutes with reduction* principle is stated as

(7)
$$\dim \mathcal{Q}(M)^\alpha = \dim \mathcal{Q}(M_\alpha),$$

where $M_\alpha$ is the reduced space of $M$ at $\alpha$. The number $\dim \mathcal{Q}(M)^\alpha$ is the **multiplicity** of the character given by $\alpha$ in $\mathcal{Q}(M)$ [**6**]. For the Dolbeault quantization, this multiplicity is 1 or 0 depending on whether $\alpha$ is in $\Delta$ or not. Since $M$ is a toric manifold, the reduced space $M_\alpha$ is a point when $\alpha$ is in $\Delta$ and is empty otherwise. Therefore, we have

$$\dim \mathcal{Q}(M) = \# \text{ of integer lattice points in } \Delta = \sum_{\alpha \in \Delta \cap \mathbb{Z}^n} \dim \mathcal{Q}(M_\alpha).$$

Counting multiplicities boils down to counting integral lattice points in $\Delta$. We can use Proposition 1 to do this count in terms of an alternating sum over shifted polarized cones.

Now, assume that $\mathbb{L}$ is a T-equivariant prequantization line bundle for $(M, \omega, \mu)$. This means that the T-action on $M$ lifts to a T-action on $\mathbb{L}$ by bundle automorphisms and also that $\mathbb{L}$ is provided with a connection $\nabla$ such that its equivariant curvature is $\omega - i\mu$. This implies that $c_1^{\mathrm{T}}(\mathbb{L}) = [\omega - i\mu]$.[2] On the other hand, by a T-equivariant version of (6) (see Bismut [**1**]), the character $\chi$ around $0 \in \mathfrak{t}$ of this representation of T on $\mathcal{Q}(M)$ is

$$(8) \qquad \chi \circ exp = \int_M \mathrm{e}^{c_1^{\mathrm{T}}(\mathbb{L})} \mathbf{Td}^{\mathrm{T}}(M),$$

where $exp$ denotes the exponential map from $\mathfrak{t}$ to T. Using the Atiyah-Bott Berline-Vergne localization formula in equivariant cohomology (c.f. [**5**]), the right hand side of (8) can be expressed as a finite sum over the fixed points $F$ of the torus action on $M$,

$$(9) \qquad \int_M \mathrm{e}^{c_1^{\mathrm{T}}(\mathbb{L})} \mathbf{Td}^{\mathrm{T}}(M) = \sum_F \mathrm{e}^{-i\langle \mu(F), u \rangle} \prod_{j=1}^n \frac{1}{1 - \mathrm{e}^{-i\langle \alpha_{j,F}, u \rangle}}.$$

Combining (9) with the notion of polarization gives us a way of counting the lattice points of the moment polytope of a toric manifold in terms of the lattice points of cones based on the vertices of the polytope. This insight provided by the localization formula in the context of quantization commutes with reduction for toric manifolds inspires us to give a weighted polar decomposition for any simple polytope, not necessarily integral. The proof of this result is completely independent from the use of the localization formula and is shown in Section §4.

## 3. Weighted quantization commutes with reduction

DEFINITION 2. [**7**] The equivariant Hirzebruch $\chi_y$-characteristic of an equivariant complex line bundle $\mathbb{L}$ over a toric manifold $M$ is

$$(10) \qquad \chi_y(M, \mathbb{L}) = \int_M \mathrm{e}^{c_1^{\mathrm{T}}(\mathbb{L})} \mathbf{Q}_y^{\mathrm{T}}(M).$$

Motivated by (9), we replace $\mathbf{Td}^{\mathrm{T}}(M)$ by $\mathbf{Q}_y^{\mathrm{T}}(M)$ in (8) and use the Atiyah-Bott Berline-Vergne localization formula again to compute (10).

Let $(M, \omega, \mu)$ be a dimension $2n$ toric manifold with moment polytope $\mu(M) = \Delta$. Assume that the vertices of $\Delta$ are in $\mathbb{Z}^n$, then $M$ is prequantizable. Let $\mathbb{L} \to M$ be a prequantization complex line bundle over $M$ and assume furthermore that $\mathbb{L}$ is T-equivariant. The equivariant Chern class $c_1^{\mathrm{T}}(\mathbb{L})$ is equal to $[\omega - i\mu]$. The

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[2]the first T-equivariant Chern class $c_1^{\mathrm{T}}(\mathbb{L})$ is of the form $[\omega + \mu]$ up to a constant. We use the convention $c_1^{\mathrm{T}}(\mathbb{L}) = [\omega - i\mu]$ for technical reasons.

complexified tangent bundle of $M$ stably splits into a direct sum of complex line bundles $\oplus L_j$ which we assume are also T-equivariant. Then the T-equivariant Hirzebruch class $\mathbf{Q}_y^{\mathrm{T}}(M)$ is

$$\mathbf{Q}_y^{\mathrm{T}}(M) = \mathbf{Td}^{\mathrm{T}}(M) \prod_{j=1}^n \frac{1}{1+y}(1 + y\mathrm{e}^{-c_1^{\mathrm{T}}(L_j)}),$$

where $y$ is any real number not equal to $-1$. By the Atiyah-Bott Berline-Vergne localization formula, we get

$$
\begin{aligned}
& \int_M \mathrm{e}^{c_1^{\mathrm{T}}(\mathbb{L})} \mathbf{Q}_y^{\mathrm{T}}(M) \\
(11) \quad &= \sum_{F \in M^T} \frac{\mathrm{e}^{c_1^{\mathrm{T}}(\mathbb{L})}|_F (\mathbf{Td}^{\mathrm{T}}(M) \prod_{j=1}^n \frac{1}{1+y}(1 + y\mathrm{e}^{-c_1^{\mathrm{T}}(L_j)}))|_F}{\mathrm{e}^{\mathrm{T}}(M)|_F} \\
&= \sum_{F \in M^T} \mathrm{e}^{c_1^{\mathrm{T}}(\mathbb{L})}|_F \prod_{j=1}^n \left( \frac{1}{1+y} \left( \frac{1 + y\mathrm{e}^{-c_1^{\mathrm{T}}(L_j)}}{1 - \mathrm{e}^{-c_1^{\mathrm{T}}(L_j)}} \right) \right) \Bigg|_F.
\end{aligned}
$$

Let $u$ be a generic element of the Lie algebra $\mathfrak{t} \cong i\mathbb{R}^n$ and $v = \mu(F)$ be the vertex of the moment polytope $\Delta$ corresponding to the fixed point $F$. For $1 \leq j \leq n$, let $\alpha_{j,v}$ be edge vectors at $v$ normalized as in Subsection §§2.1. This implies that our choice of such vectors is unique. These edge vectors can be thought of as the weights of the isotropic representation of T on the normal bundle $N_F = T_F M$ at $F$ [6]. The restriction of the equivariant Chern class $c_1^{\mathrm{T}}(L_j)$ to $F$ is $c_1^{\mathrm{T}}(L_j)|_F = i\langle \alpha_{j,v}, u \rangle$ for $1 \leq j \leq n$. Similarly, the restriction of $c_1^{\mathrm{T}}(\mathbb{L})$ to $F$ is $c_1(\mathbb{L})|_F = -i\langle v, u \rangle$. Then, (11) can be written as

$$(12) \quad \int_M \mathrm{e}^{c_1^{\mathrm{T}}(\mathbb{L})} \mathbf{Q}_y^{\mathrm{T}}(M) = \sum_v \mathrm{e}^{-i\langle v, u \rangle} \prod_{j=1}^n \left( \frac{1}{1+y} \left( \frac{1 + y\mathrm{e}^{-i\langle \alpha_{j,v}, u \rangle}}{1 - \mathrm{e}^{-i\langle \alpha_{j,v}, u \rangle}} \right) \right),$$

where the sum is over all vertices of $\Delta$. We pick $\xi \in \mathbb{R}^n \setminus \{0\}$ such that $\langle \alpha_{j,v}, \xi \rangle \neq 0$ for $1 \leq j \leq n$ and all vertices $v$ in $\Delta$. As seen on Subsection §§2.1, we can always find such a $\xi$. Letting $u = i\xi$, we get

$$
\begin{aligned}
(13) \quad &\sum_v \mathrm{e}^{-i\langle v, i\xi \rangle} \prod_{j=1}^n \left( \frac{1}{1+y} \left( \frac{1 + y\mathrm{e}^{-i\langle \alpha_{j,v}, i\xi \rangle}}{1 - \mathrm{e}^{-i\langle \alpha_{j,v}, i\xi \rangle}} \right) \right) \\
&= \sum_v \mathrm{e}^{\langle v, \xi \rangle} \prod_{j=1}^n \left( \left[ \frac{1}{1+y} + \frac{y\mathrm{e}^{\langle \alpha_{j,v}, \xi \rangle}}{1+y} \right] \frac{1}{1 - \mathrm{e}^{\langle \alpha_{j,v}, \xi \rangle}} \right).
\end{aligned}
$$

Denote $\mathrm{e}^{\langle \alpha_{j,v}, \xi \rangle}$ by $z_{j,v}$. If $\langle \alpha_{j,v}, \xi \rangle < 0$, we set $\alpha_{j,v}^\sharp = \alpha_{j,v}$ and have $z_{j,v} = \mathrm{e}^{\langle \alpha_{j,v}^\sharp, \xi \rangle}$, so that $|z_{j,v}| < 1$ and

$$\frac{1}{1 - z_{j,v}} = 1 + z_{j,v} + z_{j,v}^2 + \cdots = \sum_{m=0}^\infty z_{j,v}^m.$$

If $\langle \alpha_{j,v}, \xi \rangle > 0$, we set $\alpha_{j,v}^\sharp = -\alpha_{j,v}$ and have $z_{j,v}^{-1} = \mathrm{e}^{\langle \alpha_{j,v}^\sharp, \xi \rangle}$, so that $|z_{jv}^{-1}| < 1$ and

$$\frac{1}{1 - z_{j,v}} = -z_{j,v}^{-1} \frac{1}{1 - z_{j,v}^{-1}} = -z_{j,v}^{-1}(1 + z_{j,v}^{-1} + z_{j,v}^{-2} + \cdots) = -\sum_{m=1}^\infty z_{j,v}^{-m}.$$

Thus, when $|z_{j,v}| < 1$,

$$(14) \qquad \left(\frac{1}{1+y} + \frac{yz_{j,v}}{1+y}\right)\frac{1}{1-z_{j,v}} = \frac{1}{1+y} + \sum_{m=1}^{\infty} z_{j,v}^m,$$

whereas, if $|z_{j,v}^{-1}| < 1$,

$$(15) \qquad \left(\frac{1}{1+y} + \frac{yz_{j,v}}{1+y}\right)\frac{1}{1-z_{j,v}} = -\left(\frac{y}{1+y} + \sum_{m=1}^{\infty} z_{j,v}^{-m}\right).$$

Now, denote $e^{\langle \alpha_{j,v}^\sharp, \xi \rangle}$ by $z_{j,v}$, regardless $\alpha_{j,v}^\sharp = \alpha_{j,v}$ or $\alpha_{j,v}^\sharp = -\alpha_{j,v}$. Plugging (14) and (15) in (13), we obtain

$$(16) \qquad \sum_v (-1)^{\#v} e^{\langle v,\xi \rangle} \prod_{k=1}^{n-\#v}\left(\frac{1}{1+y} + \sum_{m_k=1}^{\infty} z_{j_k,v}^{m_k}\right)\prod_{l=1}^{\#v}\left(\frac{y}{1+y} + \sum_{m_l=1}^{\infty} z_{j_l,v}^{m_l}\right),$$

where $(-1)^{\#v}$ denotes the number of $\alpha_{j,v}$ flipped by the polarizing vector $\xi$. Setting $v = (v_1, \cdots, v_n)$, $\xi = (\xi_1, \cdots, \xi_n)$ and $\alpha_{j,v}^\sharp = (\alpha_{j,v}^{\sharp 1}, \cdots, \alpha_{j,v}^{\sharp n})$, we have

$$e^{\langle v,\xi \rangle} = e^{v_1\xi_1 + \cdots + v_n\xi_n} \quad \text{and}$$

$$z_{j_k,v}^{m_k} = e^{m_k\langle \alpha_{j_k,v}^\sharp, \xi \rangle} = e^{m_k(\alpha_{j,v}^{\sharp 1}\xi_1 + \cdots + \alpha_{j,v}^{\sharp n}\xi_n)}.$$

Let $z_j = e^{\xi_j}$. Formula (16) is then equal to

$$(17) \qquad \sum_v (-1)^{\#v} z_1^{v_1} \cdots z_n^{v_n} \prod_{k=1}^{n-\#v}\left(\frac{1}{1+y} + \sum_{m_k=1}^{\infty} z_1^{m_k\alpha_{j_k,v}^{\sharp 1}} \cdots z_n^{m_k\alpha_{j_k,v}^{\sharp n}}\right)$$
$$\prod_{l=1}^{\#v}\left(\frac{y}{1+y} + \sum_{m_l=1}^{\infty} z_1^{m_l\alpha_{j_l,v}^{\sharp 1}} \cdots z_n^{m_l\alpha_{j_l,v}^{\sharp n}}\right)$$
$$= \sum_v (-1)^{\#v} z_1^{v_1} \cdots z_n^{v_n} \sum_m w_v(m) z_1^{m_1\alpha_{1,v}^{\sharp 1} + \cdots + m_n\alpha_{n,v}^{\sharp 1}}$$
$$z_2^{m_1\alpha_{1,v}^{\sharp 2} + \cdots + m_n\alpha_{n,v}^{\sharp 2}}$$
$$\vdots$$
$$z_n^{m_1\alpha_{1,v}^{\sharp n} + \cdots + m_n\alpha_{n,v}^{\sharp n}},$$

where $m = (m_1, \cdots, m_n) \in \mathbb{Z}_{\geq 0}^n$ and $w_v(m)$ is equal to

$$(18) \qquad w_v(m) = \left(\frac{1}{1+y}\right)^{r_{1,v}(m)}\left(\frac{y}{1+y}\right)^{r_{2,v}(m)}.$$

The exponent $r_{1,v}(m)$ above is the number of $m_j$ in $m$ which correspond to $\alpha_{j,v}^\sharp = \alpha_{j,v}$ and $r_{2,v}(m)$ is the number of $m_j$ in $m$ that correspond to $\alpha_{j,v}^\sharp = -\alpha_{j,v}$. We write $z_1^{v_1} \cdots z_n^{v_n}$ as $z^v$ and the long factor in the right hand side of Equation (17) as

$$z_1^{m_1\alpha_{1,v}^{\sharp 1} + \cdots + m_n\alpha_{n,v}^{\sharp 1}} z_2^{m_1\alpha_{1,v}^{\sharp 2} + \cdots + m_n\alpha_{n,v}^{\sharp 2}} \cdots z_n^{m_1\alpha_{1,v}^{\sharp n} + \cdots + m_n\alpha_{n,v}^{\sharp n}} = z^{\sum_j m_j\alpha_{j,v}^\sharp}.$$

Note that each point $p = (p_1, \ldots, p_n)$ in $\mathbf{C}_v^\sharp \cap \mathbb{Z}^n$ can be written as $p = v + \sum_j m_j \alpha_{j,v}^\sharp$ with $m_j \in \mathbb{Z}_{\geq 0}$ for $1 \leq j \leq n$, since $\{\alpha_{j,v}^\sharp\}_{j=1}^n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. We write $z^p$ for $z_1^{p_1} \ldots z_n^{p_n}$ and set $r_{1,v}(p) = r_{1,v}(m)$ and $r_{2,v}(p) = r_{2,v}(m)$ so that $w_v(p) = w_v(m)$. If $p \notin \mathbf{C}_v^\sharp$ we set $w_v(p) = 0$. Note that $r_{1,v}(p) + r_{2,v}(p)$ is equal to the codimension of the smallest dimensional face in $\mathbf{C}_v^\sharp$ containing $p$. With this notation, we can write (17) as

$$(19) \qquad \sum_v (-1)^{\#v} \sum_{p \in \mathbf{C}_v^\sharp \cap \mathbb{Z}^n} w_v(p) z^p.$$

Let $p \in \Delta \cap \mathbb{Z}^n$ and let $\xi_1$ be a polarizing vector for $\Delta$. The point $p$ may belong to more than one polarized tangent cone. Since $p$ is fixed, we have

$$(20) \qquad \sum_v (-1)^{\#v} w_v(p) z^p = \left( \sum_v (-1)^{\#v} w_v(p) \right) z^p.$$

To determine the second sum in (20), we define $c(p)$ to be the codimension of the smallest dimensional face in $\Delta$ containing $p$. If $p$ is in the interior of $\Delta$ we have $c(p) = 0$. Polarizing with respect to $\xi_1$ (in fact, with respect to any polarizing vector), we have that the polarized tangent cones of $\Delta$ are all pointing away from $p$ except one which contains $p$ in its interior. Suppose that such cone is $\mathbf{C}_v^\sharp$ for some vertex $v$ of $\Delta$. Then $r_{1,v}(p) = 0$ and $r_{2,v}(p) = 0$ since $r_{1,v}(p) + r_{2,v}(p) = c(p) = 0$. If $c(p) > 0$, the point $p$ may belong to more than one polarized tangent cone. When $c(p) = 1$, there are two cases: the sum $\sum_v (-1)^{\#v} w_v(p)$ is equal to $\frac{1}{1+y} + zeros$ or equal to $1 - \frac{y}{1+y} + zeros$. In either case, we have $\sum_v (-1)^{\#v} w_v(p) = \frac{1}{1+y}$. In general, we can see that (after simplifications) there are basically three cases: the sum $\sum_v (-1)^{\#v} w_v(p)$ is equal to $(\frac{1}{1+y})^{c(p)} + zeros$ or equal to $(1 - \frac{y}{1+y})^{c(p)} + zeros$ or equal to $(\frac{1}{1+y})^{c(p)-1} - (\frac{1}{1+y})^{c(p)-1}(\frac{y}{1+y}) + zeros$. In any case, we have $\sum_v (-1)^{\#v} w_v(p) = (\frac{1}{1+y})^{c(p)}$. A different polarizing vector $\xi_2$ have the effect of just permuting the type of polarized tangent cones (that is, cones with no edges flipped, with one edge flipped, with two edges flipped and so on) over the vertices of $\Delta$. In conclusion, regardless the polarizing vector $\xi$, we always obtain

$$(21) \qquad \sum_v (-1)^{\#v} w_v(p) = \left( \frac{1}{1+y} \right)^{c(p)}.$$

From (17) on, our arguments apply to any regular integral polytope in $\mathbb{R}^n$. Therefore, by summing over all the points of $\Delta \cap \mathbb{Z}^n$, we have proved the following statement.

THEOREM 1. *Let $\Delta$ be a regular integral polytope in $\mathbb{R}^n$. For any polarizing vector $\xi \in \mathbb{R}^n$, we have*

$$(22) \qquad \sum_v (-1)^{\#v} \sum_{p \in \mathbf{C}_v^\sharp \cap \mathbb{Z}^n} w_v(p) z^p = \sum_{p \in \Delta \cap \mathbb{Z}^n} \left( \frac{1}{1+y} \right)^{c(p)} z^p.$$

By considering equations from (10) to (22) together with the work in between, we get

$$(23) \qquad \chi_y(M, \mathbb{L}) = \int_M e^{c_1^\mathrm{T}(\mathbb{L})} \mathbf{Q}_y^\mathrm{T}(M) = \sum_{p \in \Delta \cap \mathbb{Z}^n} \left( \frac{1}{1+y} \right)^{c(p)} z^p.$$

The equivariant Hirzebruch $\chi_y$-characteristic of an equivariant complex line bundle $\mathbb{L}$ over a toric manifold $M$ is in general a polynomial in $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, as we can see from (23). This equivariant characteristic number is not in general the virtual index of an elliptic operator on $M$. However, when $y = 0$, we have $\mathbf{Q}_y^{\mathrm{T}}(M) = \mathbf{Td}^{\mathrm{T}}(M)$ and (10) is the equivariant index of the twisted Dolbeault operator $D_{\mathbb{L}}$ mentioned in Subsection §2.3. When $y = 1$, we have $\mathbf{Q}_y^{\mathrm{T}}(M) = \mathbf{L}^{\mathrm{T}}(M)$ and (10) is in this case the equivariant index of the twisted signature operator on $M$ [**11**]. Because of this, we can interpret (23) as a weighted version of the quantization commutes with reduction principle for a toric manifold.

THEOREM 2. *Let $M$ be any toric manifold whose moment polytope is denoted by $\Delta$ and let $\alpha \in \Delta \cap \mathbb{Z}^n$ be any weight lattice of the corresponding torus action on $M$. Then*

$$(24) \qquad \chi_y(M, \mathbb{L})^\alpha = \begin{cases} \left(\frac{1}{1+y}\right)^{c(\alpha)} & \text{if } \alpha \in \Delta \\ 0 & \text{if } \alpha \notin \Delta \end{cases}.$$

PROOF. It follows from (23) and the preceding discussion. □

REMARK 1. In [**13**], Y. Tian and W. Zhang perform an analytic approach to the quantization commutes with reduction principle and get a related weighted multiplicity formula for Spin$^c$-complexes twisted by certain exterior power bundles.

## 4. Weighted polar decomposition for a simple polytope

Formula (22) is valid for any regular integral polytope in $\mathbb{R}^n$. In analogy with (2), we can write (22) using characteristic functions and extend the result to any simple polytope, not necessarily integral.

Let $\Delta$ be a simple polytope in $\mathbb{R}^n$ and let $x$ be any point in this polytope. Let $c(x)$ be the codimension of the smallest dimensional face in $\Delta$ containing $x$. We define the following weighted characteristic function over $\Delta$:

$$(25) \qquad \mathbf{1}_\Delta^w(x) = \begin{cases} \left(\frac{1}{1+y}\right)^{c(x)} & \text{if } x \in \Delta \\ 0 & \text{if } x \notin \Delta \end{cases}.$$

Given a polarizing vector $\xi$ in $\mathbb{R}^n$, let $\mathbf{C}_v^\sharp$ be the polarized cone corresponding to the vertex $v$. If $x$ is in $\mathbf{C}_v^\sharp$, let $c_v(x)$ be the codimension of the smallest dimensional face $F$ in $\mathbf{C}_v^\sharp$ containing $x$. We define $r_{1,v}(x)$ and $r_{2,v}(x)$ as before; namely, $r_{1,v}(x)$ is the number of unflipped edges emanating from $v$ which do not belong to $F$, and $r_{2,v}(x)$ is the number of flipped edges emanating from $v$ which are not in $F$. Therefore, $r_{1,v}(x) + r_{2,v}(x) = c_v(x)$. For any $y \neq -1$, we set

$$(26) \qquad w_v(x) = \left(\frac{1}{1+y}\right)^{r_1(x)} \left(\frac{y}{1+y}\right)^{r_2(x)}.$$

We define the following weighted characteristic function over $\mathbf{C}_v^\sharp$:

$$(27) \qquad \mathbf{1}_{\mathbf{C}_v^\sharp}^{w_v}(x) = \begin{cases} w_v(x) & \text{if } x \in \mathbf{C}_v^\sharp \\ 0 & \text{if } x \notin \mathbf{C}_v^\sharp \end{cases}.$$

PROPOSITION 2. *For any simple polytope $\Delta$ and for any choice of polarizing vector, we have*

$$\mathbf{1}_\Delta^w = \sum_v (-1)^{\#v} \mathbf{1}_{\mathbf{C}_v^\sharp}^{w_v}, \tag{28}$$

*where $\mathbf{1}_\Delta^w$ and $\mathbf{1}_{\mathbf{C}_v^\sharp}^{w_v}$ are the weighted characteristic functions over $\Delta$ and $\mathbf{C}_v^\sharp$ defined in (25) and (27), where the sum is over the vertices of $\Delta$, and where $\#v$ denotes the number of edge vectors at $v$ flipped by the polarizing process (1).*

The proof of this result collects the main ideas from the proof of Theorem 1. Notice that when $y = 0$, we obtain the ordinary polar decomposition stated in Proposition 1. When $y = 1$, the weighted characteristic functions of the polarized tangent cones are the same at all vertices. In this case, we can write (28) as in [**8**] and [**9**],

$$\mathbf{1}_\Delta^w = \sum_v (-1)^{\#v} \mathbf{1}_{\mathbf{C}_v^\sharp}^w .$$

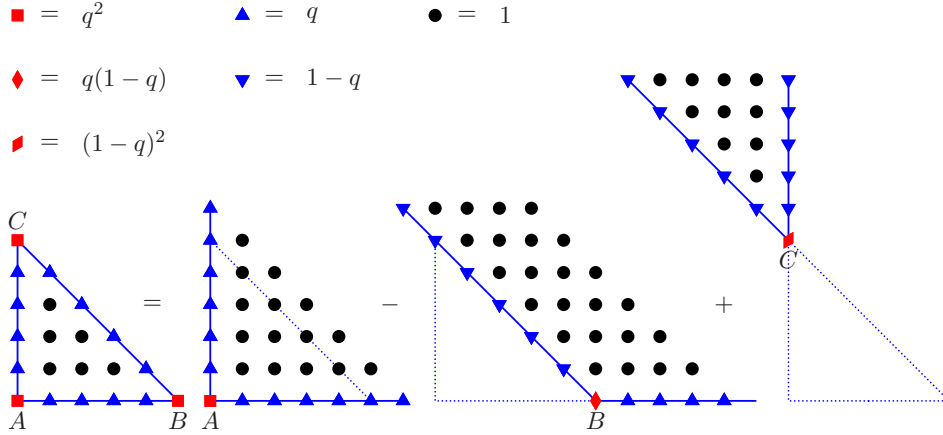We illustrate Proposition 2 for a triangle in Figure 2.



FIGURE 2. Weighted polar decomposition of a lattice triangle

*Proof.* The proof has two steps. First, for each $x \in \mathbb{R}^n$ we choose a polarizing vector $\xi_x$ such that (28) holds. For this, we show that $x$ appears in at most one polarized tangent cone in the corresponding polar decomposition of $\Delta$ given by $\xi_x$. Second, we show that the right hand side of (28) is independent of the choice of polarization.

Notice that for any polarization there is always a unique cone with no edges flipped at all. More explicitly, if $x$ is in the exterior of $\Delta$, we can take the closest vertex of $\Delta$ to $x$ (if there are more than one such a vertex, pick any one of them). Say we take $v_1$. Then, we set $\mathbf{C}_{v_1}^\sharp = \mathbf{C}_{v_1}$. Clearly $x$ is not in $\mathbf{C}_{v_1}^\sharp$. We polarize according to this cone. Namely, let $\xi_x$ be a polarizing vector in $\mathbb{R}^n$ that makes $\mathbf{C}_{v_1}^\sharp = \mathbf{C}_{v_1}$ and let $E_{\xi_x}$ be a codimension one plane with normal vector parallel to $\xi_x$. Moving this plane in a parallel way through the polytope, we flip the edges of the tangent cones whose vertices lie below $E_{\xi_x}$ (according to $\xi_x$) until we get to the farthest vertex $v_2$ away from $v_1$ in the direction of $\xi_x$. There is only one such a vertex since $\xi_x$ is a polarizing vector. The corresponding polarized tangent

cone $\mathbf{C}_{v_2}^\sharp$ has all their edges flipped with respect to $\mathbf{C}_{v_2}$ and it does not contain $x$. In general, a polarizing vector flips an edge if this edge is shared with a previous polarized tangent cone and leaves that edge unflipped otherwise. Then, for any vertex other than $v_1$ and $v_2$, the polarized tangent cones have at least one edge flipped and they all appear pointing away from $x$. We conclude that $x$ appears in no polarized tangent cone.

If $x$ is in $\Delta$, we can pick any vertex $v$ in the face $F$ whose relative interior contains $x$. Then, we set the polarized tangent cone $\mathbf{C}_v^\sharp$ equal to $\mathbf{C}_v$ and choose a polarizing vector $\xi_x$ that makes $\mathbf{C}_v^\sharp = \mathbf{C}_v$. We now repeat the process explained above. Again, all other polarized tangent cones will point away from $x$, with the cone at the farthest vertex away from $v$ having all their edges flipped.

From here on, we follow [**8**] closely with some modifications. Let

$$E_1, \cdots, E_N$$

denote all the different codimension one subspaces of $\mathbb{R}^{n*}$ that are equal to

$$\alpha_{j,v}^\perp = \{\eta \in \mathbb{R}^{n*} \,|\, \langle \eta, \alpha_{j,v}\rangle = 0\}$$

for some $j$ and $v$. If no two edges of $\Delta$ are parallel, then the number $N$ of such hyperplanes is equal to the number of edges of $\Delta$. A vector $\xi$ is a *polarizing vector* if and only if it does not belong to any $E_j$. The *polarized tangent cones* $\mathbf{C}_v^\sharp$ only depend on the connected component of the complement

$$\mathbb{R}^{n*} \setminus (E_1 \cup \cdots \cup E_N)$$

in which $\xi$ lies. Any two polarizing vectors can be connected by a path $\xi_t$ in $\mathbb{R}^{n*}$ which crosses the *walls* $E_j$ one at a time.

The second part of the proof consists of showing that the right hand side of formula (28) does not change when the polarizing vector $\xi_t$ crosses a single wall, $E_k$.

As $\xi_t$ crosses the wall $E_k$, the sign of the pairing $\langle \xi_t, \alpha_{j,v}\rangle$ changes exactly if $E_k = \alpha_{j,v}^\perp$. For each vertex $v$, denote by $S_v(x)$ and $S_v'(x)$ its contributions to the right hand side of formula (28) before and after $\xi_t$ crosses the wall. The vertices for which these contributions differ are exactly those vertices that lie on edges $e$ of $\Delta$ which are perpendicular to $E_k$. They come in pairs because each edge has two endpoints.

Let us concentrate on one such an edge, $e$, with endpoints, say, $u$ and $v$. Let $\alpha_e$ denote an edge vector at $v$ that points from $v$ to $u$ along $e$. Suppose that the pairing $\langle \xi_t, \alpha_e\rangle$ flips its sign from negative to positive as $\xi_t$ crosses the wall; (otherwise we switch the roles of $v$ and $u$). The *polarized tangent cones* to $\Delta$ at $v$, before and after $\xi_t$ crosses the wall, are

$$\mathbf{C}_v^\sharp = v + \sum_{j \in I_e} \mathbb{R}_{\geq 0}\alpha_{j,v}^\sharp + \mathbb{R}_{\geq 0}\alpha_e \quad \text{and} \quad (\mathbf{C}_v^\sharp)' = v + \sum_{j \in I_e} \mathbb{R}_{\geq 0}\alpha_{j,v}^\sharp - \mathbb{R}_{\geq 0}\alpha_e,$$

where $I_e \subset \{1, \cdots, N\}$ encodes the facets that contain $e$. (The $\alpha_{j,v}^\sharp$ are the same for the different $\xi_t$'s because the pairings $\langle \xi_t, \alpha_{j,v}^\sharp\rangle$ do not flip sign when $\xi_t$ crosses the wall for $j \in I_e$.) Notice that the cones $\mathbf{C}_v^\sharp$ and $(\mathbf{C}_v^\sharp)'$ have a common facet and their union only depends on the edge $e$ and not on the endpoint $v$. (This uses the assumption that the polytope $\Delta$ is simple and follows from the fact that $\alpha_{j,u} \in \mathbb{R}_{\geq 0}\alpha_{j,v} + \mathbb{R}\alpha_e$.)

To determine the contributions of $v$ to the right hand side of (28) before and after $\xi_t$ crosses the wall, we have two cases to analyze. In the first case, the smallest dimensional face $F$ containing $x$ also contains the edge $e$. Then, the contributions are

$$S_v(x) = \varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)} \left( \frac{y}{1+y} \right)^{r_2(x)} \quad \text{and} \quad S'_v(x) = -\varepsilon \cdot 0,$$

where $\varepsilon \in \{-1, 1\}$. Their difference is

$$S_v(x) - S'_v(x) = \varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)} \left( \frac{y}{1+y} \right)^{r_2(x)}.$$

The contributions of the other endpoint, $u$, are

$$S_u(x) = -\varepsilon \cdot 0 \quad \text{and} \quad S'_u(x) = \varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)} \left( \frac{y}{1+y} \right)^{r_2(x)},$$

and their difference is

$$S_u(x) - S'_u(x) = -\varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)} \left( \frac{y}{1+y} \right)^{r_2(x)}.$$

Notice that the numbers $r_1(x)$ and $r_2(x)$ remain the same for both endpoints because $F$ contains the edge $e$. Hence, the differences $S_v(x) - S'_v(x)$ and $S_u(x) - S'_u(x)$, for the two endpoints $u$ and $v$ of $e$, sum to zero.

In the second case, the edge $e$ is not contained in $F$. Then, the contributions are

$$S_v(x) = \varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)} \left( \frac{y}{1+y} \right)^{r_2(x)} \quad \text{and}$$

$$S'_v(x) = -\varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)-1} \left( \frac{y}{1+y} \right)^{r_2(x)+1},$$

and their difference is

$$S_v(x) - S'_v(x) = \varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)-1} \left( \frac{y}{1+y} \right)^{r_2(x)}.$$

On the other hand, after crossing the wall, the smallest dimensional face $F'$ in $\mathbf{C}_u^\sharp$ containing $x$ has codimension one less than the codimension of $F$ in $\mathbf{C}_v^\sharp$, before crossing the wall. Then, a straightforward calculation shows that the contributions of the other endpoint, $u$, are

$$S_u(x) = -\varepsilon \cdot 0 \quad \text{and} \quad S'_u(x) = \varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)-1} \left( \frac{y}{1+y} \right)^{r_2(x)},$$

and their difference is

$$S_u(x) - S'_u(x) = -\varepsilon \left( \frac{1}{1+y} \right)^{r_1(x)-1} \left( \frac{y}{1+y} \right)^{r_2(x)}.$$

Hence, the differences $S_v(x) - S'_v(x)$ and $S_u(x) - S'_u(x)$, for the two endpoints $u$ and $v$ of $e$, sum to zero again.     $\square$

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